

Hamiltonian Mechanics on Quaternion Kähler Manifolds

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Abstract

The goal of this study is to present quaternion Kähler analogue of Hamiltonian mechanics. Finally, the some results related to quaternion Kähler dynamical systems were also given.

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1 Introduction

It is well-known that modern differential geometry explains explicitly the dynamics of Hamiltonians. Therefore, if Q is an m -dimensional configuration manifold and $\mathbf{H} : T^*Q \rightarrow \mathbf{R}$ is a regular Hamiltonian function, then there is a unique vector field X on T^*Q such that dynamic equations are given by

$$i_X \Phi = d\mathbf{H} \tag{1}$$

where Φ indicates the symplectic form. The triple (T^*Q, Φ, X) are called *Hamiltonian system* on the cotangent bundle T^*Q .

Nowadays, there are a lot of studies about Hamiltonian mechanics, formalisms, systems and equations [1, 2] and there in. There are real, complex, paracomplex and other analogues. We say that in order to obtain different analogous in different spaces is possible.

Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra. It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics [3] .

In this study, therefore, Hamiltonian equations related to mechanical systems on quaternion

Kähler manifold have been presented.

2 Preliminaries

Throughout this paper, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted. $\mathcal{F}(M)$, $\chi(M)$ and $\Lambda^1(M)$ denote the set of functions on M , the set of vector fields on M and the set of 1-forms on M , respectively.

2.1 Quaternion Kähler Manifolds

Let M be an n -dimensional manifold. It has a 3-dimensional vector bundle V consisting of tensors of type (1,1). The manifold M satisfies the condition given by:

(a) In any coordinate neighborhood U of M , there exists a local basis $\{F, G, H\}$ of V such that

$$F^2 = G^2 = H^2 = FGH = -I. \quad (3)$$

I denotes the identity tensor of type (1,1) in M . Such a local basis $\{F, G, H\}$ is called a canonical local basis of the bundle V in U . Then V is said to be an almost quaternion structure in M , and M with V is an almost quaternion manifold denoted by (M, V) . An almost quaternion manifold M is of dimension $n = 4m$ ($m \geq 1$). In any almost quaternion manifold (M, V) , there is a Riemannian metric tensor field g such that

$$g(\phi X, Y) + g(X, \phi Y) = 0 \quad (4)$$

for any cross-section ϕ and any vector fields X, Y of M . An almost quaternion structure V fixed with a Riemannian metric g is called an almost quaternion metric structure. A manifold M endowed with an almost quaternion metric structure $\{g, V\}$ is said to be an almost quaternion metric manifold denoted by (M, g, V) . Let $\{F, G, H\}$ be a canonical local basis of V an almost quaternion manifold (M, g, V) . Since each of F, G and H is almost Hermitian with respect to g , setting

$$\Phi(X, Y) = g(FX, Y), \quad \Psi(X, Y) = g(GX, Y), \quad \Theta(X, Y) = g(HX, Y) \quad (5)$$

for any vector fields X and Y , we see that Φ, Ψ and Θ are local 2-forms.

Assume that the Riemannian connection ∇ of (M, g, V) satisfies the conditions as follows:

(b) If ϕ is a cross-section (local or global) of the bundle V , then $V_X \phi$ is also a cross-section of V , where X is an arbitrary vector field in M . From (3) we see that the condition (b) is equivalent to the following condition:

(b') If F, G, H is a canonical local basis of V , then

$$\nabla_X F = r(X)G - q(X)H, \quad \nabla_X G = -r(X)F + p(X)H, \quad \nabla_X H = q(X)F - p(X)G \quad (6)$$

for any vector field X , where p, q and r are certain local 1-forms. If an almost quaternion metric manifold M satisfies the condition (b) or (b'), then M is said to be a quaternion Kähler manifold and an almost quaternion structure of M is called a quaternion Kähler structure. [4]

Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1, n}$ be a real coordinate system on a neighborhood U of M , and suppose that let $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}} \right\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over \mathbf{R} of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M , respectively. Taking

into [5], then the following expression can be found

$$\begin{aligned}
F\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{n+i}}, \quad F\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{3n+i}}, \quad F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_{2n+i}} \\
G\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{2n+i}}, \quad G\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}}, \quad G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_i}, \quad G\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{n+i}} \\
H\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{3n+i}}, \quad H\left(\frac{\partial}{\partial x_{n+i}}\right) = \frac{\partial}{\partial x_{2n+i}}, \quad H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_{n+i}}, \quad H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_i}
\end{aligned} \tag{7}$$

A canonical local basis $\{F^*, G^*, H^*\}$ of V^* of the cotangent space $T^*(M)$ of manifold M satisfies the condition as follows:

$$F^{*2} = G^{*2} = H^{*2} = F^*G^*H^* = -I, \tag{8}$$

defining by

$$\begin{aligned}
F^*(dx_i) &= dx_{n+i}, \quad F^*(dx_{n+i}) = -dx_i, \quad F^*(dx_{2n+i}) = dx_{3n+i}, \quad F^*(dx_{3n+i}) = -dx_{2n+i}, \\
G^*(dx_i) &= dx_{2n+i}, \quad G^*(dx_{n+i}) = -dx_{3n+i}, \quad G^*(dx_{2n+i}) = -dx_i, \quad G^*(dx_{3n+i}) = dx_{n+i}, \\
H^*(dx_i) &= dx_{3n+i}, \quad H^*(dx_{n+i}) = dx_{2n+i}, \quad H^*(dx_{2n+i}) = -dx_{n+i}, \quad H^*(dx_{3n+i}) = -dx_i.
\end{aligned}$$

3 Hamiltonian Mechanics

Here, we obtain Hamiltonian equations and Hamiltonian mechanical system for quantum and classical mechanics structured on quaternion Kähler manifold (M, V) .

Firstly, let (M, V) be a quaternion Kähler manifold. Assume that a component of almost quaternion structure V^* , a Liouville form and a 1-form on quaternion Kähler manifold (M, V) are shown by F^* , λ_{F^*} and ω_{F^*} , respectively.

Then $\omega_{F^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i})$ and $\lambda_{F^*} = F^*(\omega_{F^*}) = \frac{1}{2}(x_i dx_{n+i} - x_{n+i} dx_i + x_{2n+i} dx_{3n+i} - x_{3n+i} dx_{2n+i})$. It is concluded that if Φ_{F^*} is a closed Kähler form on quaternion Kähler manifold (M, V) , then Φ_{F^*} is also a symplectic structure on quaternion Kähler manifold (M, V) .

Consider that Hamilton vector field X associated with Hamiltonian energy \mathbf{H} is given by

$$X = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}}. \quad (9)$$

Then

$$\Phi_{F^*} = -d\lambda_{F^*} = dx_{n+i} \wedge dx_i + dx_{3n+i} \wedge dx_{2n+i}$$

and

$$i_X \Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i} dx_i - X^i dx_{n+i} + X^{3n+i} dx_{2n+i} - X^{2n+i} dx_{3n+i}. \quad (10)$$

Moreover, the differential of Hamiltonian energy is obtained as follows:

$$d\mathbf{H} = \frac{\partial \mathbf{H}}{\partial x_i} dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}} dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} dx_{3n+i}. \quad (11)$$

According to **Eq.**(1), if equaled **Eq.** (10) and **Eq.** (11), the Hamiltonian vector field is found as follows:

$$X = -\frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{n+i}} - \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{3n+i}}. \quad (12)$$

Suppose that a curve

$$\alpha : I \subset \mathbf{R} \rightarrow M \quad (13)$$

be an integral curve of the Hamiltonian vector field X , i.e.,

$$X(\alpha(t)) = \dot{\alpha}, \quad t \in I. \quad (14)$$

In the local coordinates, it is obtained that

$$\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i}) \quad (15)$$

and

$$\dot{\alpha}(t) = \frac{dx_i}{dt} \frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt} \frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt} \frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt} \frac{\partial}{\partial x_{3n+i}}. \quad (16)$$

Considering **Eq.** (14), if equaled **Eq.** (12) and **Eq.** (16), it follows

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \quad (17)$$

Thus, the equations obtained in **Eq.** (17) are seen to be *Hamiltonian equations* with respect to component F^* of almost quaternion structure V^* on quaternion Kähler manifold (M, V) , and then the triple (M, Φ_{F^*}, X) is seen to be a *Hamiltonian mechanical system* on quaternion Kähler manifold (M, V) .

Secondly, let (M, V) be a quaternion Kähler manifold. Suppose that an element of almost quaternion structure V^* , a Liouville form and a 1-form on quaternion Kähler manifold (M, V) are denoted by G^* , λ_{G^*} and ω_{G^*} , respectively.

Then $\omega_{G^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i})$ and $\lambda_{G^*} = G^*(\omega_{G^*}) = \frac{1}{2}(x_i dx_{2n+i} - x_{n+i} dx_{3n+i} - x_{2n+i} dx_i + x_{3n+i} dx_{n+i})$. It is known if Φ_{G^*} is a closed Kähler form on quaternion Kähler manifold (M, V) , then Φ_{G^*} is also a symplectic structure on quaternion Kähler manifold (M, V) .

Let X be a Hamiltonian vector field related to Hamiltonian energy \mathbf{H} and given by **Eq.** (9).

Considering

$$\Phi_{G^*} = -d\lambda_{G^*} = dx_{2n+i} \wedge dx_i + dx_{n+i} \wedge dx_{3n+i}, \quad (18)$$

then we calculate

$$i_X \Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i} dx_i - X^i \frac{\partial}{\partial x_i} dx_{2n+i} + X^{n+i} dx_{3n+i} - X^{3n+i} dx_{n+i}. \quad (19)$$

Besides, the differential of Hamiltonian energy is as follows:

$$d\mathbf{H} = \frac{\partial \mathbf{H}}{\partial x_i} dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}} dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} dx_{3n+i}. \quad (20)$$

According to **Eq.**(1), if we equal **Eq.** (19) and **Eq.** (20), it follows

$$X = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{3n+i}}. \quad (21)$$

Considering **Eq.** (14), **Eqs.** (16) and (21) are equal, we find equations

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}} \quad (22)$$

In the end, the equations obtained in **Eq.** (22) are known to be *Hamiltonian equations* with respect to component G^* of almost quaternion structure V^* on quaternion Kähler manifold (M, V) , and then the triple (M, Φ_{G^*}, X) is a *Hamiltonian mechanical system* on quaternion Kähler manifold (M, V) .

Thirdly, let (M, V) be a quaternion Kähler manifold. By H^* , λ_{H^*} and ω_{H^*} , we denote a component of almost quaternion structure V^* , a Liouville form and a 1-form on quaternion Kähler manifold (M, V) , respectively.

Then $\omega_{H^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i})$ and $\lambda_{H^*} = H^*(\omega_{H^*}) = \frac{1}{2}(x_i dx_{3n+i} + x_{n+i} dx_{2n+i} - x_{2n+i} dx_{n+i} - x_{3n+i} dx_i)$. It is well-known that if Φ_{H^*} is a closed Kähler form on quaternion Kähler manifold (M, V) , then Φ_{H^*} is also a symplectic structure on quaternion Kähler manifold (M, V) .

Consider X . It is Hamiltonian vector field connected with Hamiltonian energy \mathbf{H} and given by **Eq.** (9).

Taking into

$$\Phi_{H^*} = -d\lambda_{H^*} = dx_{3n+i} \wedge dx_i + dx_{2n+i} \wedge dx_{n+i}, \quad (23)$$

we find

$$i_X \Phi_{F^*} = \Phi_{F^*}(X) = X^{3n+i} dx_i - X^i dx_{3n+i} + X^{2n+i} dx_{n+i} - X^{n+i} dx_{2n+i}. \quad (24)$$

Furthermore, the differential of Hamiltonian energy is

$$d\mathbf{H} = \frac{\partial \mathbf{H}}{\partial x_i} dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}} dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} dx_{3n+i}. \quad (25)$$

According to **Eq.**(1), **Eqs.** (24) and (25) are equaled, we obtain a Hamiltonian vector field given by

$$X = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_i} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{3n+i}}. \quad (26)$$

Taking into **Eq.** (14), we equal **Eq.** (16) and **Eq.** (26), it yields

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i} \quad (27)$$

Finally, the equations obtained in **Eq.** (27) are obtained to be *Hamiltonian equations* with respect to component H^* of almost quaternion structure V^* on quaternion Kähler manifold (M, V) , and then the triple (M, Φ_{H^*}, X) is a *Hamiltonian mechanical system* on quaternion Kähler manifold (M, V) .

4 Conclusion

Formalism of Hamiltonian mechanics has intrinsically been described with taking into account the basis $\{F^*, G^*, H^*\}$ of almost quaternion structure V^* on quaternion Kähler manifold (M, V) .

Hamiltonian models arise to be a very important tool since they present a simple method to describe the model for mechanical systems. In solving problems in classical mechanics, the rotational mechanical system will then be easily usable model.

Since physical phenomena, as well-known, do not take place all over the space, a new model for dynamic systems on subspaces is needed. Therefore, equations (17), (22) and (27) are only considered to be a first step to realize how quaternion geometry has been used in solving problems in different physical area.

For further research, the Hamiltonian vector fields derived here are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics.

References

- [1] M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, vol.152, Elsevier, Amsterdam, 1989.
- [2] M. Tekkoyun, On Para-Euler–Lagrange and Para-Hamiltonian Equations , Phys. Lett. A, Vol. 340, Issues 1-4, 2005, pp. 7-11
- [3] D. Stahlke, Quaternions in Classical Mechanics, Phys 621.
<http://www.stahlke.org/dan/phys-papers/quaternion-paper.pdf>
- [4] K. Yano, M. Kon, Structures on Manifolds, Series in Pure Mathematics-Volume 3, World Scientific Publishing Co. Pte. Ltd., Singore, 1984.
- [5] I. Burdujan, Clifford Kähler Manifolds, Balkan Journal of Geometry and its Applications, Vol.13, No:2, 2008, pp.12-23